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Nonlinear finite element formulation of the soil structure interaction through two parameter foundation model

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NONLINEAR FINITE ELEMENT FORMULATION OF THE SOIL STRUCTURE INTERACTION THROUGH TWO PARAMETER FOUNDATION MODEL

by

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To my professor and guru, Dr. Ashraf Ayoub, for his continued support and inspiration

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ABSTRACT

The response of shallow and raft foundations is having a significant importance due to its complex behavior because of the semi-infinite soil media. Winkler's model is the simplest model to deal with the structure and soil. The Winkler model represents the foundation reaction as proportional to the soil displacement at a particular point, which results in the elasticity of the soil being the only parameter in consideration. But in reality the soil cohesiveness is having a significant contribution in soil structure interaction, and therefore the consideration of coupling effects of Winkler springs need to be accounted. Most of the existing elements either consider certain parameters of the foundation or assume an elastic beam and foundation response. In this research a new finite element formulation was developed in which these limitations were eliminated. This improved model can be viewed as a soil with a combination of cohesive behavior which transmits the rotation due to bending in addition to the Winkler effect.

The non linear response of structures resting on this improved foundation model can be analyzed by assuming that the foundation resists compression and tension. In reality soil is very weak in tension and its tension capacity needs to be neglected, which leads to lift-off regions at different locations. This phenomenon becomes much more complicated by considering the inelastic soil structure behavior, which leads to a highly nonlinear problem. In order to estimate the necessary nonlinear soil parameters, an analytical procedure based on the Vlasov model is proposed. The presented solutions and applications show the superiority of the proposed nonlinear foundation model.

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1. Inelastic analysis of semi-infinite foundation elements

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Abstract

The inelastic response of shallow and raft foundations are significantly complex due to the behavior of the surrounding semi-infinite soil media. The Winkler approach models the soil as a single layer, and assumes that the foundation reaction at a particular point is proportional to the soil displacement. In reality, the soil is a semi-infinite medium that can not be modeled as a single layer. In this paper a new finite element formulation was developed in which the soil can be viewed as a semi-infinite inelastic element that can resist bending, in addition to the well-known Winkler effect. A parametric analysis of an inelastic reinforced concrete foundation element is presented.

Key Words: Beam on foundation; Vlasov foundation; Semi-infinite foundation; Mixed finite element; Hellinger-Reissner variational principle.

1. Introduction

The inelastic response of shallow and raft foundations are significantly complex due to the behavior of the surrounding semi-infinite soil media. Winkler (1867) developed a simple model that accounts for the behavior of both the foundation and soil. The Winkler model represents the soil beneath the foundation as a system of similar but mutually independent elastic springs. In this model, it is assumed that the foundation reaction at a particular point is proportional to the soil displacement. The Winkler model is considered therefore a single parameter model with the spring's elasticity as its only parameter. While this model is associated with closely spaced elastic springs, in reality these springs should be dependant on each other. To address these drawbacks, some modified

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approaches have been proposed such as the model developed by Vlasov (1966), which belongs to the family of multiple-parameter foundation models.

The stiffness matrix of a beam on multiple-parameter foundation element can be derived based on different orders of displacement shape functions or by using the exact displacement function obtained from the solution of the differential equations governing the behavior. Biot (1937) studied the foundation as an elastic continuum and gave an exact solution to an infinite beam under a concentrated load. Kerr (1965) studied the foundation response using an elastic continuum approach by connecting each two spring layers with an interconnecting shear layer. Harr et al. (1969) analyzed elastic beams on semi-infinite elastic foundations based on Vlasov general variational method. Yang (1972) introduced a numerical iterative procedure on the basis of the finite element method for analyzing plates on elastic foundations. Zhaohua and Cook (1983) developed the finite element formulation of an elastic beam on two-parameter foundation using both, an exact displacement function and a cubic displacement function for the case of distributed loads acting along the entire beam length. Chiwanga and Valsangkar (1988) extended the approach for the case of a generalized distributed load. Their work though was limited to a specific combination of beam and foundation stiffnesses. Razaqpur and Shah (1991) further extended the work by considering beams and foundations with any stiffness. Shirima and Giger (1990) developed the stiffness matrix and nodal-action column vectors for a Timoshenko beam on a two-parameter foundation element. Morfidis and Avramidis (2002) derived the element stiffness matrix based on the exact solution of the differential equations with the ability to account for shear deformations, semi-rigid connections, and rigid end offsets. In most practical applications the foundation is typically assumed to be tensionless not elastic. Kaschiev and Mikhajlov (1995) used the finite element method as a general numerical technique to solve the problem of elastic beams on tensionless foundations for different loading conditions. Coskun (2003) studied the tensionless Pasternak formulation which results in lift off regions between the beam and foundation, and presented the roots of a nonlinear equation to calculate the contact length of the beam. Celep and Demir (2005) studied the tensionless behavior which showed that the problem becomes highly non-linear due to the lift off of the beam from the foundation.

In all previously described work, the beam was assumed elastic, while the foundation was assumed either elastic or tensionless. In reality, beams are typically made up of reinforced concrete, which can undergo deformation in the plastic range due to reinforcement yielding or concrete crushing. The analysis of inelastic beam on foundation elements is very complex due to the combined effect of beam plasticity as well as the spatial variation of soil resistance. To solve this problem displacement formulation requires more elements and also are plagued by instability problems. Ayoub and Filippou (2000) and Ayoub (2003) proposed a consistent mixed formulation based on a Hellinger-Reissner variational principle for inelastic analysis of composite structures and inelastic beams on Winkler's foundations, respectively. The mixed formulation proved to overcome most of the difficulties associated with the standard displacement approach derived from a minimum potential energy principle (Zhaohua and Cook, 1983), and to provide a more efficient numerical platform for analysis of these types of structures. In this paper, the mixed approach was formulated for the problem of beams resting on semi-infinite foundations by adopting a Vlasov approach, in which the soil parameters are determined based on a plane strain approach.

In the next sections, Hellinger-Reissner finite element formulation for beams on Vlasov foundations are developed. The model is implemented in the finite element program FEAP, developed by R.L. Taylor, and described in details in Zienkiewicz and Taylor (1989). Numerical examples that compare the behavior of both models are then performed, and conclusions based on these results are derived. The governing equations of beams on Vlasov foundations are presented first.

2. Governing Equations of Beams on Vlasov Foundations

The equilibrium of an element of length *dx* of a beam element resting on a Vlasov foundation as shown in Fig. 1 is given by:

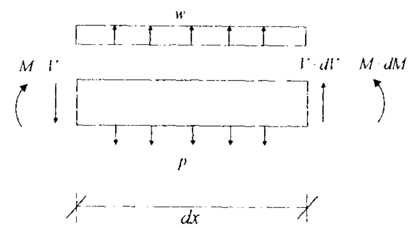

Fig. 1. Infinitesimal segment of a beam on Vlasov foundation

$$V_{,x} + (w - p) = 0$$
$$M_{,x} + V = 0 \tag{1}$$

Where V, and M denote the shear force and bending moment, respectively; p is the total foundation load per unit length acting on the beam, w denotes the distributed load on the beam, and a comma denotes derivation. According to Vlasov's hypothesis, and assuming linear soil behavior, the foundation load per unit length is related to the transverse displacement as follow:

$$p = k_f v(x) - k_m v_{,xx}(x) \tag{2}$$

Where v is the vertical displacement of the beam, k_f is the Winkler's modulus and k_m is Vlasov's parameter that depends on both the soil and foundation characteristics. For inelastic behavior, both parameters are based on nonlinear functions as will be described later. The foundation force term corresponding to Vlasov's parameter can be viewed as an additional moment resistance provided by the foundation following elementary thin-plate theories. Accordingly, the Winkler force t_f, and Vlasov force t_m which can be viewed as a moment applied to the beam by foundation, are defined respectively as follows:

$$t_f = k_f v$$
$$t_m = k_m v_{,x} \tag{3}$$

From Eqs. (1)–(3):

$$M_{,xx} - t_{m,x} + t_f - w = 0 \tag{4}$$

The values of the two foundation parameters k_f and k_m are evaluated as proposed by Vlasov (1966). These equations account for the semi-infinite dimension of the underlying soil by adopting a plane strain approach. In general the soil beneath the foundation has different stratums with different thicknesses and different soil properties. As a result, the normal stress in the soil changes with respect to the depth. Vlasov derived an equation that represents the different layer properties with a single equivalent layer with elastic modulus and Poisson ratio of E_0 and ν_0, respectively. The normal stress is assumed to be constant within this equivalent layer. According to Vlasov analysis, Zhaohua and Cook (1983) evaluated the parameters for foundations of infinite depth as follow:

$$k_f = \frac{\gamma E_0 b}{2A(1-\nu_0^2)} \text{ and } k_m = \frac{E_0 b A}{4\gamma(1+\nu_0)} \tag{5}$$

where $$A = \left[\frac{2D(1-\nu_0^2)}{E_0 b}\right]^{1/3} \text{ and } D = \frac{E_b b h^3}{12(1-\nu_b^2)} \tag{6}$$

The elastic constants of the equivalent layer E_0 and ν_0 are defined as:

$$E_0 = \frac{E_s}{1-\nu_s^2} \text{ and } \nu_0 = \frac{\nu_s}{1-\nu_s} \tag{7}$$

γ is a coefficient that depends on the elastic properties of the foundation and determines the rate of decrease of displacements over the foundation depth; b and h are the width and height of the beam section, respectively. E_b and E_s are the elastic moduli of the beam and soil, respectively and υ_b and υ_s are the Poisson ratios of the beam and soil, respectively.

The curvature at a section x is related to the transverse displacements by:

$$v_{,xx} - \chi = 0 \tag{8}$$

where χ is the section curvature.

The internal moment of the beam $M(x)$ is related to the curvature χ by a nonlinear constitutive relation

$$M(x) = \hat{g}(\chi(x)) \tag{9}$$

In this study the nonlinear relation in Eq. (9) is derived from a discretization of the cross section of the beam into several fibers with nonlinear uniaxial stress-strain relations for the constituent materials. For a nonlinear soil rather than the elastic one assumed by Vlasov, the two foundation forces t_l and t_m are related to their respective deformations by two other nonlinear relations as follows:

$$t_l = \hat{g}_l\left(v\right) \;\&\; t_m = \hat{g}_m\left(v_{,x}\right) \tag{10}$$

In the next sections, the strong form Eqs. (1)–(4) and (8)–(10) are solved for using the finite element method. Due to the nonlinear nature of Eqs. (9) and (10), a Newton-Raphson iteration strategy is used. The following discussion refers to a single Newton-Raphson iteration denoted by subscript *i*.

3. Hellinger-Reissner Formulation of Beam on Vlasov Foundation

In a Hellinger-Reissner formulation, the differential equations are solved based on both a displacement and a force field (mixed formulation). For the foundation problem, Ayoub (2003) proved earlier that this mixed approach is very advantageous from a numerical standpoint. Accordingly:

$$v(x) = \mathbf{a}(x)\mathbf{v} \tag{11}$$

In addition $$M(x) = \mathbf{b}(x)\mathbf{M} \tag{12}$$

where $M(x)$ is the bending moment, and $\mathbf{b}(x)$ is a matrix of n_m force interpolation functions, and $\mathbf{M}$ is the vector of element end moments.

The finite element formulation is derived by considering the following two variational equations that correspond to the compatibility Eq. (8) and the equilibrium Eq. (4):

$$\delta I_M = \int_\Omega \delta M^T(x)\left[v_{,xx} - \chi\right]d\Omega = 0 \qquad \Omega = \left[0, L\right] \tag{13}$$

$$\delta I_v = \int_\Omega \delta v^T(x)\left[M_{,xx} + t_l - t_{m,x} - w\right]d\Omega = 0 \qquad \Omega = \left[0, L\right] \tag{14}$$

Using an incremental Newton-Raphson iterative technique to solve the nonlinear Eq. (13):

$$\delta I_M^i = \delta I_M^{i-1} + \frac{\partial}{\partial \mathbf{M}}\delta I_M^{i-1} d\mathbf{M}^i = 0 \tag{15}$$

The left hand side of Eq. (15) is:

$$\delta I_M^i = \int_\Omega \delta M^i(x)\left[v_{,xx}^i - \chi^i\right] d\Omega = 0 \qquad \Omega = [0, L] \tag{16}$$

Substituting the predefined force interpolation functions into Eq. (16):

$$\delta I_M^i = \delta \mathbf{M}^i \int_\Omega \mathbf{b}^i(x)\left[v_{,xx}^i - \chi^i\right] d\Omega = 0 \quad \Omega = [0, L] \tag{17}$$

The second term in Eq. (15) is equal to:

$$\begin{aligned}\frac{\partial}{\partial \mathbf{M}} \delta I_M^{i-1} d\mathbf{M}^i &= \left[\delta \mathbf{M}^i \left(\int_\Omega \mathbf{b}(x)^i \frac{\partial v_{,xx}(x)}{\partial \mathbf{v}} \frac{\partial \mathbf{v}}{\partial \mathbf{M}} \right|^{i-1} d\Omega - \int_\Omega \mathbf{b}(x)^i \frac{\partial \chi(x)}{\partial \mathbf{M}(x)} \right|^{i-1} d\Omega) \right] d\mathbf{M}^i \\ &= \left[\delta \mathbf{M}^i \left(\int_\Omega \mathbf{b}(x)^i \mathbf{a}_{,xx}(x) \frac{\partial \mathbf{v}}{\partial \mathbf{M}} \right|^{i-1} d\Omega - \int_\Omega \mathbf{b}(x)^i (\hat{g}^{i-1})^{i-1}(x)\mathbf{b}(x) d\Omega) \right] d\mathbf{M}^i\end{aligned}$$

$$\Omega = [0, L] \tag{18}$$

Eq. (18) could be written as:

$$\frac{\partial}{\partial \mathbf{M}} \delta I_M^{i-1} d\mathbf{M}^i = \delta \mathbf{M}^i \left(\int_\Omega \mathbf{b}(x)^i \mathbf{a}_{,xx}(x) d\Omega \right) d\mathbf{v}^i - \delta \mathbf{M}^i \left(\int_\Omega \mathbf{b}(x)^i (\hat{g}^{i-1})^{i-1}(x)\mathbf{b}(x) d\Omega \right) d\mathbf{M}^i$$

$$\Omega = [0, L] \tag{19}$$

The first term in Eq. (15) is equal to:

$$\begin{aligned}\delta I_M^{i-1} &= \delta \mathbf{M}^i \left[-\int_\Omega \mathbf{b}^i(x) \chi^{i-1}(x) d\Omega + \int_\Omega \mathbf{b}^i v_{,xx}^{i-1}(x) d\Omega \right] \\ &= \delta \mathbf{M}^i \left[-\int_\Omega \mathbf{b}^i(x) \chi^{i-1}(x) d\Omega + \int_\Omega \mathbf{b}^i \mathbf{a}_{,xx}(x) d\Omega \mathbf{v}^{i-1} \right]\end{aligned} \qquad \Omega = [0, L] \tag{20}$$

Substituting Eqs. (19) and (20) into Eq. (15) and using the arbitrariness of $\delta\mathbf{M}$ yields:

$$\mathbf{T} d\mathbf{v}^i - \mathbf{F}^{i-1} d\mathbf{M}^i - \mathbf{v}_r^{i-1} = 0 \tag{21}$$

where
$$\mathbf{T} = \int_\Omega \mathbf{b}^i(x)\mathbf{a}_{,xx}(x) d\Omega \qquad \Omega = [0, L] \tag{22}$$

$$\mathbf{F}^{i-1} = \int_{\Omega} \mathbf{b}^{T}(x) f^{i-1}(x) \mathbf{b}(x) d\Omega, \; f(x) = \hat{g}'^{-1}(x) = k^{-1}(x), \qquad \Omega = [0, L] \qquad (23)$$

$$\mathbf{v}_r^{i-1} = \int_{\Omega} \mathbf{b}^{T}(x) \chi^{i-1}(x) d\Omega - \mathbf{T}\mathbf{v}^{i-1} \qquad \Omega = [0, L] \qquad (24)$$

$\mathbf{F}$ is the element flexibility matrix, and $\mathbf{v}_r$ is the element residual deformation vector. Similarly, using an incremental Newton-Raphson technique to solve the nonlinear Eq. (14):

$$\delta I_v^i = \delta I_v^{i-1} + \frac{\partial}{\partial \mathbf{v}} \delta I_v^{i-1} d\mathbf{v}^i = 0 \qquad (25)$$

The second term of the right hand side of Eq. (25) is equal to:

$$\frac{\partial}{\partial \mathbf{v}} \delta I_v^{i-1} d\mathbf{v}^i = \left[\delta \mathbf{v}^T \left(\int_{\Omega} \mathbf{a}(x)_{,xx}^T \mathbf{b}(x) \frac{\partial \mathbf{M}(x)}{\partial \mathbf{v}} \right|^{i-1} d\Omega + \int_{\Omega} \mathbf{a}(x)^T \left. \frac{\partial t_f(x)}{\partial \mathbf{v}} \right|^{i-1} + \int_{\Omega} \mathbf{a}(x)_{,x}^T \frac{\partial t_m(x)}{\partial v'(x)} \left. \frac{\partial v'(x)}{\partial \mathbf{v}} \right|^{i-1} d\Omega \right) \right] d\mathbf{v}^i$$

$$= \delta \mathbf{v}^T \left[\int_{\Omega} \mathbf{a}(x)_{,xx}^T \mathbf{b}(x) d\Omega \right] d\mathbf{M}(x)^i + \left[\int_{\Omega} \mathbf{a}(x)^T \hat{g}_f'^{i-1}(x) \mathbf{a}(x) d\Omega + \int_{\Omega} \mathbf{a}(x)_{,x}^T \hat{g}_m'^{i-1}(x) \mathbf{a}(x)_{,x} d\Omega \right] d\mathbf{v}^i$$

$$\Omega = [0, L] \qquad (26)$$

The first term of the right hand side of Eq. (25) is equal to:

$$\delta I_v^{i-1} = \delta \mathbf{v}^T \left[\int_{\Omega} \mathbf{a}(x)_{,xx}^T \mathbf{b}(x) \mathbf{M}^{i-1}(x) d\Omega + \int_0^L \mathbf{a}(x)^T t_f^{i-1}(x) d\Omega + \int_{\Omega} \mathbf{a}(x)_{,x}^T t_m^{i-1}(x) d\Omega + BT \right]$$

$$\Omega = [0, L] \qquad (27)$$

Substituting Eqs. (26) and (27) in Eq. (25), and from the arbitrariness of $\delta \mathbf{v}$, we get:

$$\mathbf{T}^T d\mathbf{M}^i + \mathbf{K}_f^{i-1} d\mathbf{v}^i + \mathbf{K}_m^{i-1} d\mathbf{v}^i = \mathbf{P} - \mathbf{T}^T \mathbf{M}^{i-1} - \mathbf{M}_f^{i-1} - \mathbf{M}_m^{i-1} \qquad (28)$$

Where $\mathbf{K}_f^{i-1} = \int_0^L \mathbf{a}^T(x) k_f^{i-1}(x) \mathbf{a}(x) dx$ is the Winkler foundation element stiffness matrix, $k_f(x) = \hat{g}_f'(x)$ is the foundation force tangent stiffness term,

$\mathbf{K}_m^{i-1} = \int_0^L \mathbf{a}_{,x}^T(x) k_m^{i-1}(x) \mathbf{a}_{,x}(x) dx$ is the Vlasov foundation element stiffness matrix.

$k_m(x) = \hat{g}'_m(x)$ is the foundation moment tangent stiffness term, $\mathbf{M}_f^{i-1} = \int_0^L \mathbf{a}^T(x) t_f^{i-1}(x) dx$ is the Winkler foundation element resisting load vector, $\mathbf{M}_m^{i-1} = \int_0^L \mathbf{a}_{,x}^T(x) t_m^{i-1}(x) dx$ is the Vlasov foundation element resisting load vector, $\mathbf{T}$ is as defined before in Eq. (22), and $\mathbf{P}$ is the vector of applied external loads accounting for the boundary term BT.

Writing Eqs. (21) and (28) in matrix form:

$$\begin{bmatrix} -\mathbf{F}^{i-1} & \mathbf{T} \\ \mathbf{T}^T & \mathbf{K}_f^{i-1} + \mathbf{K}_m^{i-1} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{M}^i \\ \Delta\mathbf{v}^i \end{bmatrix} = \begin{bmatrix} \mathbf{v}_r^{i-1} \\ \mathbf{P} - \mathbf{T}^T\mathbf{M}^{i-1} - \mathbf{M}_f^{i-1} - \mathbf{M}_m^{i-1} \end{bmatrix} \tag{29}$$

It is important to note that at convergence, the residual deformation vector $\mathbf{v}_r$ reduces to zero inside each element satisfying compatibility. As discussed by Ayoub (2001), two algorithms for the mixed formulation exist. In the first algorithm, the system of equations in Eq. (29) is solved for globally with the displacements and moments as degrees of freedom. This algorithm, however, results in oscillations of results. In the second algorithm, the moment degrees of freedom are condensed out from the first of Eqs. (29) at the element level resulting in a generalized displacement stiffness matrix. Accordingly:

$$\left[\mathbf{T}^T\left(\mathbf{F}^{i-1}\right)^{-1}\mathbf{T} + \mathbf{K}_f^{i-1} + \mathbf{K}_m^{i-1}\right]\Delta\mathbf{v}^i = \mathbf{P} - \mathbf{T}^T\mathbf{M}^{i-1} - \mathbf{M}_f^{i-1} - \mathbf{M}_m^{i-1} + \mathbf{T}^T\left(\mathbf{F}^{i-1}\right)^{-1}\mathbf{v}_r^{i-1} \tag{30}$$

An internal element iteration is required in order to zero the residual deformation vector $\mathbf{v}_r$ in every element. In addition, in accordance with the Babuska-Brezzi (B-B) stability condition (1973, 1974), the order of the displacement interpolation functions needs to be larger by two than that of the force interpolation functions. The algorithm used follows the same procedure as the one discussed in Ayoub (2001). Numerical examples to evaluate the performance of the model are presented next.

4. Evaluation of Model by Numerical Studies

4.1. Inelastic Beam on Tensionless Vlasov Foundation

The numerical example represents reinforced concrete beams resting on tensionless foundations, which are quite common in structural design. These primary and secondary beams are useful to connect different columns such that the load will be transferred to the soil more evenly. The reinforced concrete beam, shown in Fig. 2(a), has a length of 10 m, and a rectangular cross-section with a width and depth of 400 mm and 500 mm, respectively. The beam is having 4 No.10 (32.26 mm diameter) longitudinal bars at the top and bottom. These bars are uniformly spaced along the width with a concrete cover of 50mm . No. 3 (9.53 mm diameter) steel stirrups are provided with a spacing of 200 mm. The concrete compressive strength is assumed to be 42.0 MPa (6 ksi). Each beam section is discretized into 16 fibers with uniaxial stress-strain relations for the constituent materials, as discussed in Eq. (9). The concrete stress-strain behavior followed the Kent and Park model (1971) (Fig. 2(b)), and assuming its Young's modulus to be E_b=25113 MPa and its Poisson ratio to be ν_b=0.2. The steel's stress-strain curve is assumed to be elastic perfectly plastic with a yield stress of 413MPa (60 ksi) and a Young's modulus of 200,000 MPa (29,000 ksi). The underlying soil is Ottawa Sand with properties as given by Park and Desai (2006) as follows: elastic modulus E_s=193 MPa, Poisson ratio ν_s=0.4, and γ =1.5. The soil behavior is assumed tensionless and elasto-plastic in compression. From Eqs. (5)–(7) the values of the foundation elastic parameters were found to be: k_f =113.2 MPa and k_m=10,076,000 N. A central vertical load is applied proportionally under displacement control. A mesh consisting of 62 displacement-based elements with fifth order polynomials was shown to represent the converged solution.

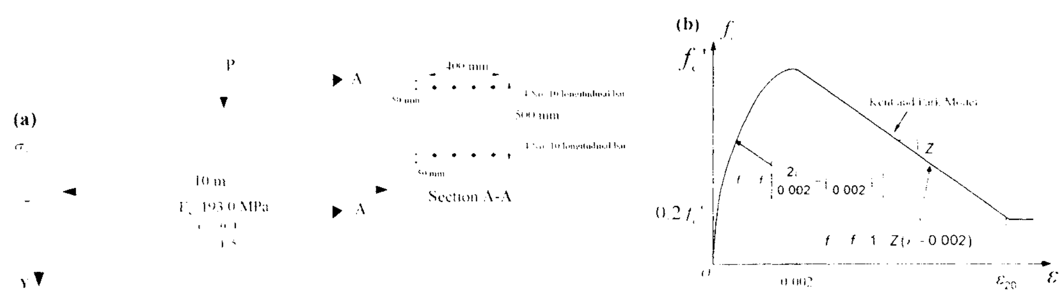

Fig. 2. (a) Reinforced concrete beam on tensionless foundation; (b) Kent and Park model for concrete.

Figs. 3 – 10 show the behavior of both the displacement and mixed models using 8 elements with different order of interpolation functions. Figs. 3 and 4 show the load - midspan displacement behavior of the Vlasov beam using the displacement and mixed formulations, respectively. For the displacement formulation, two shape functions were used, namely cubic hermitian polynomials and fifth order polynomials. For the mixed model, two force interpolation functions were used, namely linear bending moment functions along with the hermitian displacement functions; and cubic moment functions along with the fifth order displacement functions. In the figures points A, A' and A'' represent yield points for the converged, higher order and lower order solutions, respectively. Similarly, points B, B' and B'' represent the corresponding points for the ultimate state. The plots reveal that the mixed model can capture the yielding point more accurately, while having a higher convergence rate than the displacement model. In fact, for the displacement model, little difference could be observed between using the low order hermitian functions and the fifth order polynomials, while 14 mixed elements with cubic moments were sufficient to reach convergence. The superiority of the mixed model could be explained by considering the distribution of the local parameters along the length of the beam, namely the curvature, bending moment, and foundation forces. These parameters are shown in Figs. 5 – 10 for both the displacement and mixed models.

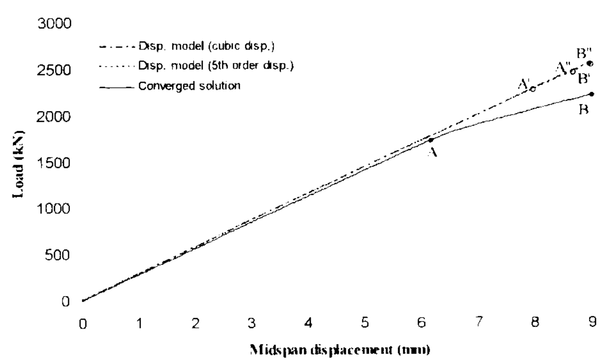

Fig. 3. Load-displacement response of RC beam on tensionless Vlasov foundation (disp. model)

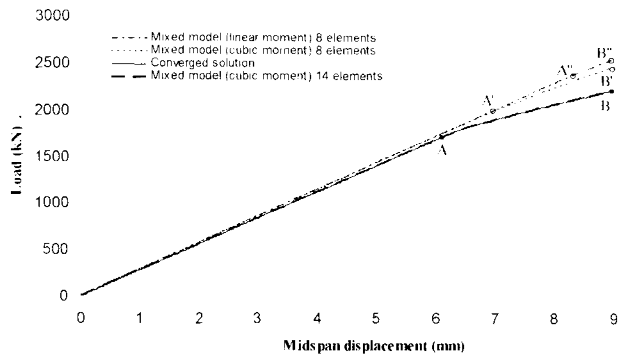

Fig. 4. Load-displacement response of RC beam on tensionless Vlasov foundation (mixed model)

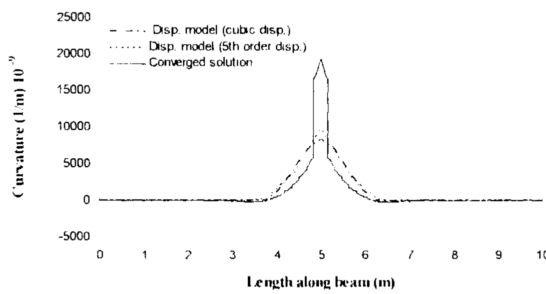

Fig. 5. Curvature distribution of RC beam on tensionless Vlasov foundation (disp. model)

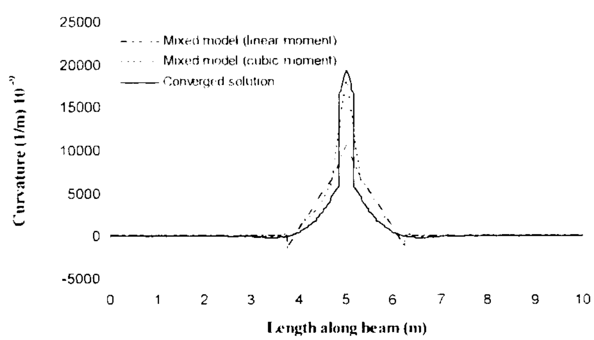

Fig. 6. Curvature distribution of RC beam on tensionless Vlasov foundation (mixed model)

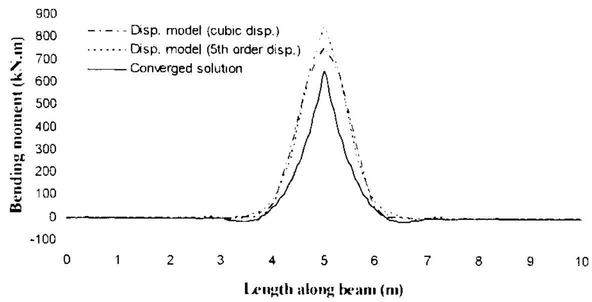

Fig. 7. Moment distribution of RC beam on tensionless Vlasov foundation (disp. model)

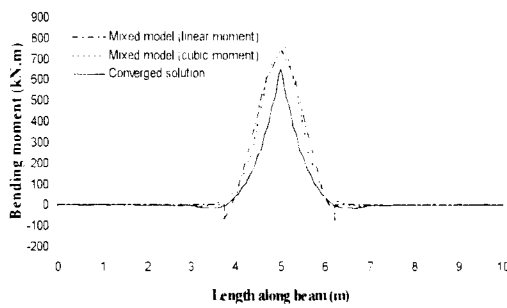

Fig. 8. Moment distribution of RC beam on tensionless Vlasov foundation (mixed model)

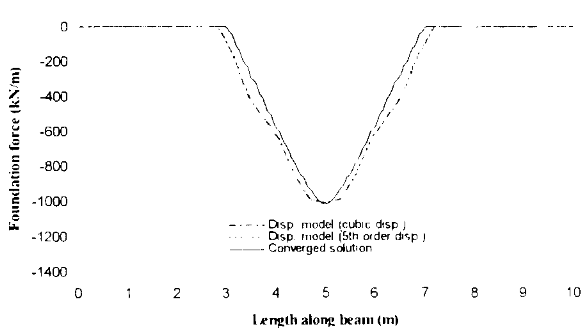

Fig. 9. Foundation force distribution of RC beam on tensionless Vlasov foundation (disp. model)

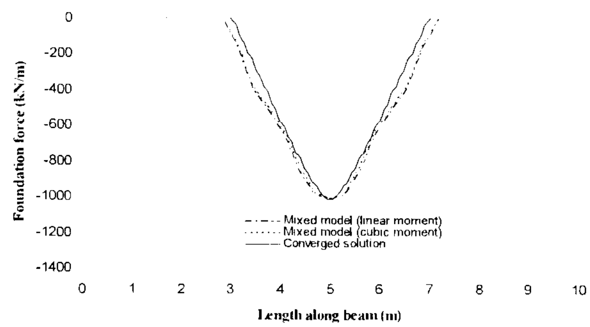

Fig. 10. Foundation force distribution of RC beam on tensionless Vlasov foundation (mixed model)

Figs. 5 and 7 show the curvature and bending moment distributions for the displacement model using both the hermitian cubic polynomials and fifth order polynomials. Due to the high value of the bending moment at midspan, the curvature is localized in this region as shown in Fig. 5. The displacement model can not capture this phenomenon accurately, even if using the higher order formulation, since it is based on polynomial displacement shape functions. Fig. 9 shows the foundation force distribution of the Vlasov beam with the displacement model.

The same plots were repeated for the mixed model. Figs. 6 and 8 show the curvature and bending moment distributions, respectively. Since the mixed model is based on approximating the smooth bending moment through the force interpolation functions, it can accurately represent the curvature localization in the plastic zone, as shown in Fig. 6. In fact, the higher order mixed formulation with cubic moment distributions produce results almost identical to the converged solution. The foundation vertical force is shown in Fig. 10. The deflected shape, as well as the lift off at the beam ends where the foundation force vanishes, was captured rather well.

4.2. Numerical Correlation with Experimental Results

Numerical analysis using the mixed model was conducted for the Aluminum shear wall foundation structure SSG04-06 tested by Gajan et al. (2006) under an increasing wall lateral load. The footing is 2.8 m x 0.65 m, and has a Young's modulus of 70,000 MPa. The underlying soil is Nevada sand with modulus of elasticity 45 MPa, Poisson ratio 0.4. The yield of the soil is assumed to be at 35% of its bearing capacity. From Eqs.

(5)– (7) the values of the foundation parameters were found to be: γ =1.5, k_f=13.5 MPa and k_m=12,170,000 N. Fig. 11 shows the monotonic envelope of the moment-rotation plot at the middle of the foundation for both the Vlasov and well-known Winkler models, as well as the experimental results. From the figure, it is observed that the Vlasov model is able to predict the behavior reasonably well, while the Winkler model under-predicts the foundation moment capacity.

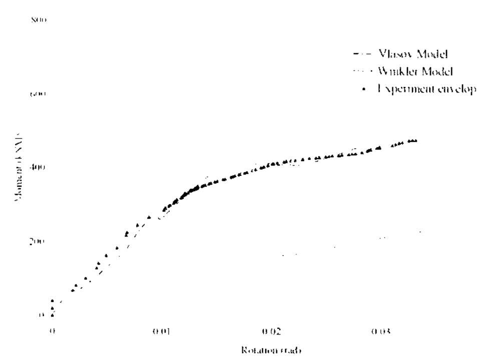

Fig. 11. Footing moment-rotation response of Gajan et al. specimen SSG04-06

5. Conclusions

This paper presents a new inelastic element for the analysis of semi-infinite foundation problems. The element is derived from a two-field mixed formulation, where forces and deformations are approximated with independent interpolation functions. The nonlinear response of the foundation is analyzed following a Vlasov approach to represent the semi-infinite soil medium. Numerical examples to evaluate the performance of the model were conducted. The studies revealed the superiority of the proposed mixed model in evaluating the inelastic complex behavior of these types of structures.

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2. Nonlinear Finite Element Modeling of Beams on Two-Parameter Foundations

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Abstract

This paper presents an inelastic element for the analysis of beams resting on two-parameter foundations. The element is derived from a two field mixed formulation with independent approximation of forces and displacements. The values for the two parameters of the foundation are derived through an iterative technique that is based on an assumption of plane strain for the soil medium. This iterative behavior is repeated at each time step of the nonlinear solution algorithm. The nonlinear response of structures resting on this improved two-parameter foundation model is analyzed following both a Vlasov and a Pasternak approach. Numerical examples that clarify the advantage of the newly developed model are conducted. These studies confirmed the importance of accounting for the foundation second parameter, and the efficiency and accuracy of the proposed model.

Key Words: Two-parameter foundation; Winkler foundation; Pasternak foundation; Vlasov foundation; Mixed finite element.

1. Introduction

The inelastic response of shallow and raft foundations is significantly complex due to the behavior of the surrounding semi-infinite soil media. Winkler's model [1] is the simplest element that account for the behavior of both the foundation and soil. The Winkler approach models the soil as a single layer, and assumes that the foundation reaction at a particular point is proportional to the soil displacement. The Winkler model is considered therefore a single-parameter model with the spring's elasticity as its only parameter. While this model is associated with closely spaced independent elastic

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springs, in reality these springs should be dependant on each other. To address these drawbacks, several modified approaches have been proposed such as the ones developed by Filonenko-Borodich [2], Hetenyi [3], Pasternak [4], and Vlasov and Leontiev [5]. These models belong to the family of multiple-parameter foundation models because, in addition to the first parameter spring's elasticity, they assumed a second parameter that accounts for the effect of the adjacent soil medium. In this paper, the Pasternak [4], and Vlasov and Leontiev [5] models were adopted. In the Pasternak model, the shear interaction between the Winkler spring elements is considered. The shear layers consist of incompressible vertical elements which deform only in transverse shear. In the Vlasov and Leontiev model, the second parameter is considered by extending the Pasternak spring elements with a consideration of the effect of the soil on both sides of the beam.

The stiffness matrix of an elastic beam on multiple-parameter foundation element can be derived based on different orders of displacement shape functions or by using the exact displacement function obtained from the solution of the differential equations governing the behavior. Biot [6] studied the foundation as an elastic continuum and derived an exact solution to an infinite beam under a concentrated load. Kerr [7] studied the foundation response using an elastic continuum approach by connecting each two spring layers with an interconnecting shear layer. Reissner [8] formulated the problem of an elastic plate on an elastic foundation by assuming a transition condition at the interior of the foundation layer along the cylindrical surface. Harr et al. [9] analyzed beams on elastic foundations based on Vlasov general variational method in which the elastic foundation is represented by a single layer. Yang [10] introduced a numerical iterative procedure on the basis of the finite element method for analyzing plates on elastic foundations. Zhaohua and Cook [11] developed the finite element formulation of an elastic beam on two-parameter foundation using both, an exact displacement function, and a cubic displacement function for the case of distributed loads acting along the entire beam length. Chiwanga and Valsangkar [12] extended the approach for the case of a generalized distributed load. Shirma and Giger [15] developed the stiffness matrix and nodal-action column vectors for a Timoshenko beam on two-parameter foundation element. Razaqpur and Shah [14] derived the stiffness matrix and nodal load vector of an element representing a beam on two-parameter elastic foundation using polynomial displacement shape functions. Vallabhan and Das [15] developed a unique iterative technique to determine the values of the Vlasov parameters used in [9]. De Rosa [16] studied the free vibration of Timoshenko beams on two-parameter foundations by considering the second parameter either as the total rotation of the beam or the bending rotation only. Hou and Tseng [17] used cubic polynomial expressions for the total deflection and bending slope of the beam, and derived the mass and stiffness matrices of the element from energy expressions. Patel et al. [18] further modified the formulation of beam on two-parameter elements by considering geometric nonlinearity with Von Karman's strain-displacement relations. Onu [19] presented a formulation with an explicit free of meshing stiffness matrix with the consideration of shear deformation effects. Morfidis and Avramidis [20] derived the element stiffness matrix based on the exact solution of the differential equations with the ability to account for shear deformations, semi-rigid connections, and rigid offsets. Coskun [21] studied the tensionless Pasternak formulation which results in lift-off regions between the beam and foundation, and presented the roots of a nonlinear equation to calculate the contact length of the beam. In most practical applications though, the foundation is typically assumed to be tensionless not elastic. Celep and Demir [22] studied the tensionless behavior of beams resting on Winkler foundations, which showed that the problem becomes highly nonlinear due to the lift-off of the beam from the foundation. Kaschiev and Mikhajlov [23] used the finite element method as a general numerical technique to solve the problem of elastic beams on tensionless foundations for different loading conditions. Ayoub and Filippou [24] and Ayoub [25] proposed a consistent mixed formulation for inelastic analysis of composite structures. The mixed formulation proved to overcome most of the difficulties associated with the standard displacement approach and to provide a more efficient numerical platform for analysis of these types of structures. Ayoub [26] confirmed the advantages of the mixed formulation over displacement-based formulations for beams on Winkler foundations.

The objective of this paper is to adopt the numerically efficient mixed formulation for developing a new element for inelastic analysis of beams resting on two-parameter foundations. In the next sections, both a displacement and a mixed finite element formulation for beams on two-parameter foundations are developed. The models are implemented in the finite element program FEAP, developed by R.L. Taylor, and described in details in Zienkiewicz and Taylor [27]. Numerical examples that compare the behavior of both models are then performed, and conclusions based on these results are derived. The governing equations of beams on two-parameter foundations are presented first.

2. Governing Equations for Beams on Two-Parameter Foundations

2.1. Equilibrium

The equilibrium of an element of length dx of a beam element resting on a two-parameter foundation, as shown in Fig. (1), is given by:

$$V_{,x} + \left(w - t_f \right) = 0 \tag{1}$$

$$M_{,x} + V = 0 \tag{2}$$

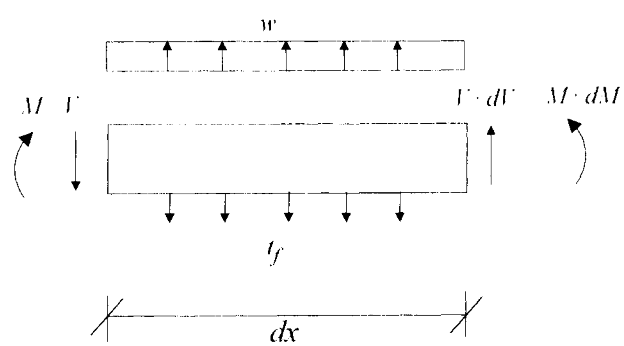

Fig. 1. Infinitesimal segment of a beam on two-parameter foundation

Where V and M denote the shear force and bending moment, respectively. t_f is the foundation force per unit length, w denotes the distributed load on the beam, and a comma denotes a derivative. According to the two-parameter foundation hypothesis, and assuming linear soil behavior, the foundation force per unit length is related to the transverse displacement as follow:

$$t_f = k_f v(x) - k_m v_{,xx}(x) \tag{3}$$

Where k_f is the Winkler's modulus and k_m is a second parameter that depends on both the soil and foundation characteristics. For inelastic behavior, both parameters will be based on nonlinear functions as will be described later. The foundation force term corresponding to the second parameter can be viewed as an additional moment resistance provided by the foundation following elementary thin-plate theories. Accordingly, the foundation moment per unit length, assuming linear behavior, is defined as follows:

$$t_m = k_m v_{,x} \tag{4}$$

From equations (1) to (3):

$$M_{,xx} - t_{m,x} + t_f - w = 0 \tag{5}$$

The values of the two foundation parameters k_f and k_m are typically evaluated based on two-parameter equations [5]. For an approximate analysis, Vlasov and Leontiev [5] assumed the transverse displacement $\bar{v}(x, y)$ as a function of a vertical surface displacement $v(x)$ and a shape function $h(y)$. These equations were derived for a beam of finite width resting on an elastic foundation layer in a plane strain condition, as shown in Fig. 2 :

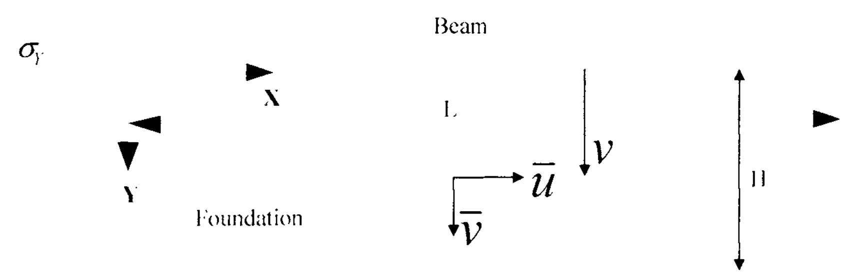

Fig. 2. Beam on elastic foundation

$$\bar{v}(x, y) = v(x)h(y) \tag{6}$$

Where $h(0) = 1$, $h(H) = 0$: H being the depth of the soil layer, and

$$h(y)=\frac{\sinh\ \gamma\left(1-\frac{y}{H}\right)}{\sinh\ \gamma} \tag{7}$$

Based on these assumptions, the parameters k_f and k_m are evaluated as:

$$k_f=\frac{(1-\nu_s)E_s b}{(1+\nu_s)(1-2\nu_s)H}\frac{\gamma\sinh\gamma\cosh\gamma+\gamma^2}{2\sinh^2\gamma} \tag{8}$$

$$k_m=\frac{E_s bH}{2(1+\nu_s)}\frac{\sinh\gamma\cosh\gamma-\gamma}{2\gamma\sinh^2\gamma} \tag{9}$$

The γ parameter is a coefficient that determines the rate of decrease of the displacements over the depth of the foundation and is evaluated from the equation given below [5]:

$$\left(\frac{\gamma}{H}\right)^2=\frac{(1-2\nu_s)}{2(1-\nu_s)}\frac{\int_0^L\left(\frac{dv}{dx}\right)^2dx+0.5\sqrt{\frac{k_f}{k_m}}\left(v^2(0)+v^2(L)\right)}{\int_0^L v^2(x)dx+0.5\sqrt{\frac{k_m}{k_f}}\left(v^2(0)+v^2(L)\right)} \tag{10}$$

Where E_s , ν_s are elastic modulus and Poisson ratio of the soil, respectively and L is the soil length. For the Vlasov model, an additional soil length, typically taken as twice the beam length on both of its sides is assumed. This length was proven to be sufficient to capture the semi-infinite soil effect [20]. In the Pasternak model, this additional soil length is ignored.

To determine the parameter γ an iterative method developed by Vallabhan and Das [15] is adopted as follows: First assume a value of γ and calculate the k_f and k_m parameters form Eqs. (8) and (9), respectively. With these parameters computed, the new surface displacement $v(x)$ is evaluated. The parameter γ is then recalculated from Eq. (10), and compared to the previously evaluated γ value. The process is repeated at each time step of the nonlinear solution algorithm until convergence is achieved within an acceptable tolerance.

2.2. Compatibility

The curvature at a section x is related to the transverse displacements by:

$$v_{,xx} - \chi = 0 \tag{11}$$

where v is the vertical displacement of the beam, and χ is the curvature.

2.3. Material Constitutive Laws

The internal moment of the beam *M(x)* is related to the curvature χ by a nonlinear constitutive relation

$$M(x) = \hat{\mathbf{g}}\left(\chi(x)\right) \tag{12}$$

In this study the nonlinear relation in (12) is derived from a fiber discretization of the cross section of the beam with nonlinear uniaxial stress-strain relations for the constituent materials. The two foundation forces t_f and t_m are related to their respective deformations by two other nonlinear relations as follows:

$$t_f = \hat{g}_f\left(v\right) \ \& \ t_m = \hat{g}_m\left(v'\right) \tag{13}$$

In the next sections, the strong form equations (1) to (5) and (11) to (13) are solved for using the finite element method. Due to the nonlinear nature of equations (12) and (13), a Newton-Raphson iteration strategy is used. The following discussion refers to a single Newton-Raphson iteration denoted by subscript *i*.

3. Displacement Formulation of Beam on Two-Parameter Foundation

In a displacement formulation, the differential equations are solved based on a displacement field. Accordingly:

$$v(x) = \mathbf{a}(x)\mathbf{V} \tag{14}$$

where $v(x)$ is the vertical displacement, and $\mathbf{a}(x)$ is a matrix of n_d shape functions, n_d depends on the order of displacement shape functions, and $\mathbf{V}$ is the vector of element end displacements.

The finite element formulation is considered by deriving the weighted integral of the equilibrium equation:

$$\int_L \delta v^T(x)\left[M_{,xx} - t_{m,x} + t_f - w\right]dx = 0 \tag{15}$$

where , denotes derivation.

Integrating by parts twice the first term and once the third term, and ignoring the distributed load term w:

$$\int_L \delta v(x)_{,xx}^T M dx + \int_L \delta v_{,x}^T(x) t_m dx + \int_L \delta v^T(x) t_f dx + BT = 0 \tag{16}$$

where the Boundary term BT equals to

$$BT = \delta v M_{,x}\Big|_0^L - \delta v_{,x} M\Big|_0^l - \delta v t_{m,x}\Big|_0^l$$

The consistent linearization of the nonlinear force-deformation relation for the beam and foundation yield:

$$\begin{aligned} M^i &= k^{i-1}\Delta v_{,xx}{}^i + M^{i-1} \\ t_m^i &= k_m^{i-1}\,\Delta v_{,x}{}^i + t_m^{i-1} \\ t_f^i &= k_f^{i-1}\,\Delta v^i + t_f^{i-1} \end{aligned} \tag{17}$$

Where k, k_m, k_f are the derivatives of the nonlinear functions $\hat{\mathbf{g}}$, $\hat{\mathbf{g}}_m$, and $\hat{\mathbf{g}}_f$.

Substituting (17) into (16):

$$\int_0^l \delta v_{,xx}{}^T(x)\left[k^{i-1}\Delta v_{,xx}{}^i + M^{i-1}\right]dx + \int_0^l \delta v_{,x}{}^T(x)\left[k_m^{i-1}\,\Delta v_{,x}{}^i + t_m^{i-1}\right]dx + \int_0^l \delta v^T(x)\left[k_f^{i-1}\,\Delta v^i + t_f^{i-1}\right]dx = BT \tag{18}$$

Substituting the predefined displacement shape functions $\mathbf{a}(x)$ into (12). we get:

$$\begin{aligned} &\delta\mathbf{v}^T\left[\int_0^l \mathbf{a}_{,xx}^T(x)\, k^{i-1}\mathbf{a}_{,xx}\, dx + \int_0^l \mathbf{a}_{,x}{}^T(x) k_m^{i-1}\,\mathbf{a}_{,x}\, dx + \int_0^l \mathbf{a}^T(x) k_f^{i-1}\,\mathbf{a}\, dx\right]\Delta\mathbf{v}^i \\ &= \delta\mathbf{v}^T\left[\mathbf{P}\text{-} \int_0^l \mathbf{a}_{,xx}^T(x)\, M^{i-1} dx - \int_0^l \mathbf{a}_{,x}{}^T(x) t_m^{i-1}\, dx - \int_0^l \mathbf{a}^T(x) t_f^{i-1}\, dx\right] \end{aligned} \tag{19}$$

From the arbitrariness of $\delta \mathbf{v}$, we get:

$$\left(\mathbf{K}^{i-1} + \mathbf{K}_m^{i-1} + \mathbf{K}_f^{i-1}\right) d\mathbf{v}^i = \mathbf{P} - \mathbf{M}^{i-1} - \mathbf{M}_m^{i-1} - \mathbf{M}_f^{i-1} \tag{20}$$

where

$\mathbf{K}^{i-1} = \int_0^L \mathbf{a}_{,xx}^T(x) \boldsymbol{k}^{i-1}(x) \mathbf{a}_{,xx}(x) dx$, is the beam element stiffness matrix,

where $\boldsymbol{k}(x)$ is the beam section stiffness term

$\mathbf{K}_f^{i-1} = \int_0^L \mathbf{a}^T(x) k_f^{i-1}(x) \mathbf{a}(x) dx$, is the Winkler foundation element stiffness matrix,

where $k_f(x)$ is the foundation force stiffness term

$\mathbf{K}_m^{i-1} = \int_0^L \mathbf{a}_{,x}^T(x) k_m^{i-1}(x) \mathbf{a}_{,x}(x) dx$, is the two-parameter foundation element stiffness matrix,

where $k_m(x)$ is the foundation moment stiffness term

$\mathbf{M}^{i-1} = \int_0^L \mathbf{a}_{,xx}^T M^{i-1}(x) dx$, is the beam element resisting load vector

$\mathbf{M}_f^{i-1} = \int_0^L \mathbf{a}^T(x) t_f^{i-1}(x) dx$, is the Winkler foundation element resisting load vector

$\mathbf{M}_m^{i-1} = \int_0^L \mathbf{a}_{,x}^T(x) t_m^{i-1}(x) dx$, is the two-parameter foundation element resisting load vector

and $\mathbf{P}$ is the vector of applied external loads.

4. Mixed Formulation of Beam on Two-Parameter Foundation

In a two-field mixed formulation, the differential equations are solved based on both a displacement and a force field. For the foundation problem, it was proven earlier that this mixed approach is very advantageous from a numerical standpoint [26]. Accordingly, and similar to (14):

$$v(x) = \mathbf{a}(x)\mathbf{V} \tag{21}$$

In addition

$$M(x) = \mathbf{b}(x)\mathbf{M} \tag{22}$$

Where $M(x)$ is the bending moment, and $\mathbf{b}(x)$ is a matrix of n_m force interpolation functions, and $\mathbf{M}$ is the vector of element end moments.

The finite element formulation is considered by deriving the weighted integral forms of both the compatibility and equilibrium equations:

$$\int_{\Omega} \delta M^{T}(x)\left[v_{,xx} - \chi\right]dx = 0 \tag{23}$$

$$\int_{\Omega} \delta v^{T}(x)\left[M_{,xx} - t_{m,x} + t_{f} - w\right]dx = 0 \tag{24}$$

The incremental section constitutive law is inverted an substituted in (23). Accordingly:

$$\chi^{i} = f^{i-1}\Delta M^{i} + \chi^{i-1} \tag{25}$$

$$\int_{0}^{L} \delta M^{T}(x)\left[v_{,xx}{}^{i}(x) - f^{i-1}\Delta M^{i} - \chi^{i-1}\right]dx = 0 \tag{26}$$

where f^{i-1} is the section flexibility term at the previous Newton-Raphson iteration.

Substituting the predefined displacement shape functions and force interpolation functions into the weak form (26), we get:

$$\delta \mathbf{M}^{T}\left\{\left[\int_{0}^{L} \mathbf{b}^{T}(x)\mathbf{a}_{,xx}(x)\,dx\right]\mathbf{v}^{i} - \left[\int_{0}^{L} \mathbf{b}^{T}(x) f^{i-1}(x)\mathbf{b}(x)\,dx\right]\Delta\mathbf{M}^{i} - \int_{0}^{L} \mathbf{b}^{T}(x)\chi^{i-1}(x)dx\right\} = 0 \tag{27}$$

From the arbitrariness of $\delta\mathbf{M}$, we get:

$$\left[\int_{0}^{L} \mathbf{b}^{T}(x)\mathbf{a}_{,xx}(x)\,dx\right]\mathbf{v}^{i} - \left[\int_{0}^{L} \mathbf{b}^{T}(x) f^{i-1}(x)\mathbf{b}(x)\,dx\right]\Delta\mathbf{M}^{i} - \int_{0}^{L} \mathbf{b}^{T}(x)\chi^{i-1}(x)dx = 0 \tag{28}$$

Substituting $\mathbf{v}^{i}$ by $\mathbf{v}^{i-1} + \Delta\mathbf{v}^{i}$, (28) becomes:

$$\mathbf{T}\,\Delta\mathbf{v}^{i} - \mathbf{F}^{i-1}\Delta\mathbf{M}^{i} - \mathbf{v}_{\mathbf{r}}{}^{i-1} = 0 \tag{29}$$

where

$$\mathbf{T} = \int_0^L \mathbf{b}^T(x)\,\mathbf{a}_{,xx}(x)\mathrm{d}x\,,\quad \mathbf{F}^{i-1} = \int_0^L \mathbf{b}^T(x) f^{i-1}(x)\,\mathbf{b}(x)\mathrm{d}x\,,\quad \mathbf{v}_r^{\,i-1} = \int_0^L \mathbf{b}^T(x)\,\chi^{i-1}(x)\,dx - \mathbf{T}\,\mathbf{v}^{i-1} \tag{30}$$

where $\mathbf{F}$ is the element flexibility matrix, and $\mathbf{v}_r$ is the element residual deformation vector.

Integrating by parts twice the first term and once the second term in (24), we get:

$$\int_0^L \delta v_{,xx}^T(x)\, M^i(x)\,dx + \int_0^L \delta v_{,x}^{\;T}(x)\, t_m^i(x)\,dx + \int_0^L \delta v^T(x)\, t_f^i(x)\,dx = \text{Boundary Terms (BT)} \tag{31}$$

The incremental force-deformation relations of the foundation take the form

$$t_f^i = k_f^{i-1}\,\Delta v^i + t_f^{i-1} \text{ and } t_m^i = k_m^{i-1}\,\Delta v^i + t_m^{i-1} \tag{32}$$

Substituting (32) into (31) results in

$$\int_0^L \delta v_{,xx}^{\;T}(x)\, M^i(x)\,dx + \int_0^L \delta v_{,x}^{\;T}(x)\left[k_m^{i-1}\,\Delta v^i + t_m^{i-1}\right]dx + \int_0^L \delta v^T(x)\left[k_f^{i-1}\,\Delta v^i + t_f^{i-1}\right]dx = \mathrm{BT} \tag{33}$$

Substituting the predefined displacement shape functions and force interpolation functions into the weak form (33), we get:

$$\delta\mathbf{v}^T \left\{\left[\int_0^L \mathbf{a}_{,xx}^T(x)\,\mathbf{b}(x)\,dx\right]\mathbf{M}^i + \int_0^L \mathbf{a}_{,x}^{\;T}(x)\left[k_m^{i-1}\,\Delta v^i + t_m^{i-1}\right]dx + \int_0^L \mathbf{a}^T(x)\left[k_f^{i-1}\,\Delta v^i + t_f^{i-1}\right]dx\right\} = \mathrm{BT} \tag{34}$$

From the arbitrariness of $\delta\mathbf{v}$ and replacing $\mathbf{M}^i = \mathbf{M}^{i-1} + \Delta\mathbf{M}^i$, (34) is rewritten as:

$$\mathbf{T}^T\,\Delta\mathbf{M}^i + \mathbf{K}_f^{i-1}\Delta\mathbf{v}^i + \mathbf{K}_m^{i-1}\Delta\mathbf{v}^i = \mathbf{P} - \mathbf{T}^T\,\mathbf{M}^{i-1} - \mathbf{M}_f^{i-1} - \mathbf{M}_m^{i-1} \tag{35}$$

where $\mathbf{K}_f^{i-1}$, $\mathbf{K}_m^{i-1}$, $\mathbf{M}_f^{i-1}$, $\mathbf{M}_m^{i-1}$ are as defined in (14), $\mathbf{T}$ is as defined before in (30), and $\mathbf{P}$ is the vector of applied external loads.

Writing equations (29) and (35) in matrix form:

$$\begin{bmatrix} -\mathbf{F}^{i-1} & \mathbf{T} \\ \mathbf{T}^{t} & \mathbf{K}_{f}^{i-1}+\mathbf{K}_{m}^{i-1} \end{bmatrix}\begin{bmatrix} \Delta\mathbf{M}^{i} \\ \Delta\mathbf{v}^{i} \end{bmatrix}=\begin{bmatrix} \mathbf{v}_{r}^{i-1} \\ \mathbf{P}-\mathbf{T}^{t}\mathbf{M}^{i-1}-\mathbf{M}_{f}^{i-1}+\mathbf{M}_{m}^{i-1} \end{bmatrix} \tag{36}$$

It is important to note that at convergence, the residual deformation vector $\mathbf{v}_r$ reduces to zero inside each element satisfying compatibility. As discussed by Ayoub and Filippou [24], two algorithms for the mixed formulation exist. In the first algorithm, the system of equations in (36) is solved for globally with the displacements and moments as degrees of freedom. This algorithm, however, results in oscillations of results. In the second algorithm, the moment degrees of freedom are condensed out from the first of equations (36) at the element level resulting in a generalized displacement stiffness matrix. Accordingly:

$$\left[\mathbf{T}^{t}\left(\mathbf{F}^{i-1}\right)^{-1}\mathbf{T}+\mathbf{K}_{f}^{i-1}+\mathbf{K}_{m}^{i-1}\right]\Delta\mathbf{v}^{i}=\mathbf{P}-\mathbf{M}^{i-1}-\mathbf{M}_{f}^{i-1}-\mathbf{M}_{m}^{i-1}+\mathbf{T}^{t}\left(\mathbf{F}^{i-1}\right)^{-1}\mathbf{v}_{\mathbf{r}}^{i-1} \tag{37}$$

An internal element iteration is required in order to zero the residual deformation vector $\mathbf{v}_{\mathbf{r}}$ in every element. The algorithm used follows the same procedure as the one discussed in Ayoub [25].

4.1. Stability of Mixed Formulation

The order and continuity of stress and displacement interpolation functions are very important parameters in a mixed formulation. For stability of the formulation the rank of matrix T in the expression $\mathbf{T}^{t}\left(\mathbf{F}^{i-1}\right)^{-1}\mathbf{T}$ in Eq. (37) should not be larger than the rank of the flexibility matrix $\mathbf{F}$ for the limit case where the foundation stiffness matrix is zero. For this to be the case the number of unknowns $n_d^{'}$ in vector $\mathbf{v}$ after excluding their rigid body modes should be less or equal to the number of unknowns n_s in vector $\mathbf{M}$:

$$n_s \geq n_d^{'} \tag{38}$$

While condition (38) is necessary for stability of the problem, there is no accuracy gain by increasing the order of the force field beyond that of the deformation field that respects the strain – displacement compatibility condition. The equality condition of (38), i.e. $n_s = n_d^{'}$ is therefore the most efficient choice from a computational standpoint. As a result, the Babuska-Brezzi (B-B) stability conditions [28-29] for the beam on two-parameter foundation element states that the order of the displacement interpolation functions needs to be larger by two than that of the force interpolation functions.

5. Evaluation of Model by Numerical Studies

5.1. Elastic Beam on Vlasov Foundation

The proposed two-parameter model with Pasternak and Vlasov effect (effect of the soil on either side of the beam is considered) was evaluated by analyzing a beam of finite length resting on an elastic foundation, which was first studied by Shirma and Ginger [13].

The beam length is 5m, width b=0.4m and depth h=1.0m. The beam is made out of timber with an elastic modulus E_b=10,500 MPa and Poisson ratio $\nu_b = 0.25$. The elastic foundation is sandy clay with an elastic modulus E_s=45.4 MPa, Poisson ratio $\nu_s = 0.25$ and $\gamma = 1.0$. From equations (8) to (10), the values of the foundation parameters are: k_f = 3.081 MPa and k_m =12,449,000 N.

The beam has free ends and is subjected to a concentrated moment of 50 kNm applied at the center as shown in Fig. (3). The beam is discretized into 4 mixed elements with cubic moment interpolation functions. Five integration points were assumed for each finite element. Since the problem is elastic, the load is applied at mid span under load control. To evaluate the effect of the semi-infinite foundation, the beam is analyzed using a Winkler, Pasternak, and Vlasov formulation. Fig. (4) shows the midspan moment rotation behavior of all models. From the figure, at the highest moment value of 50 kN.m, the Winkler rotation was found to be 2.8 times larger than the Vlasov rotation. Fig. (5) shows the deflected shape of the models. Because of the added rotational resistance due to the Vlasov parameter, the end deflection of the Winkler foundation is about three times that of the Vlasov foundation. Furthermore, due to the effect of the soil on both sides of the beam in the Vlasov model, its end deflection is 2.1 times less than that of the Pasternak model. The bending moment is slightly underestimated if ignoring the Vlasov or Pasternak effects, as shown in Fig. (6).

The same beam with free ends is analyzed assuming the foundation to be tensionless, as shown in Fig. (7). An axial force P that equals 100 kN is applied under load control.

while a moment M is applied incrementally under displacement control. Fig. (8) shows the midspan moment rotation behavior of the beam for the Vlasov and Winkler foundations, respectively using a mixed model with cubic moment interpolation functions. The Vlasov foundation moment resistance is found to be about 2.8 times larger than that of the Winkler foundation moment resistance at a rotation of 0.01 rad. Fig. (9) shows the foundation vertical displacement of both models at the ultimate load, which reveals that the Winkler model is having slightly larger deformations than the Vlasov model. From Fig. (10), which shows the foundation rotation for both models at the ultimate load, the Winkler model has a rotation 11% higher than that of the Vlasov model. The preceding discussion confirms the need to account for the semi-infinite soil effects in analyzing beam on foundation problems.

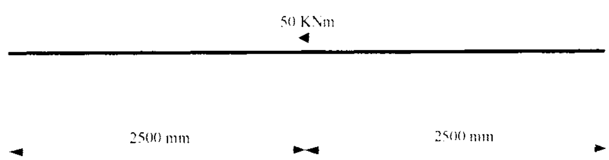

Fig. 3. Beam with free ends

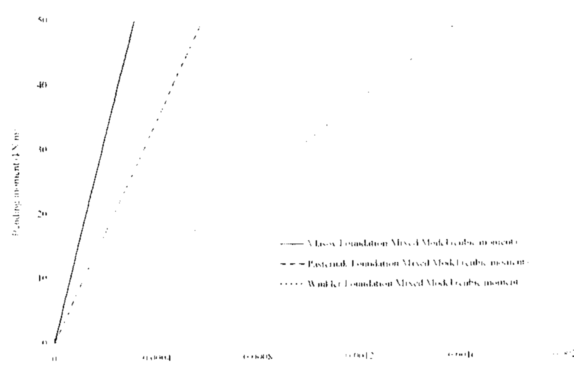

Fig. 4. Moment-rotation response of free beam

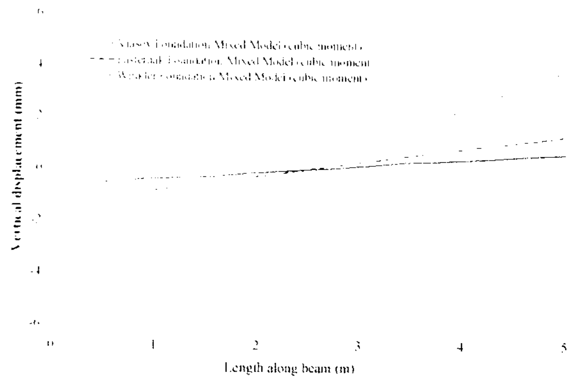

Fig. 5. Vertical displacement for free beam

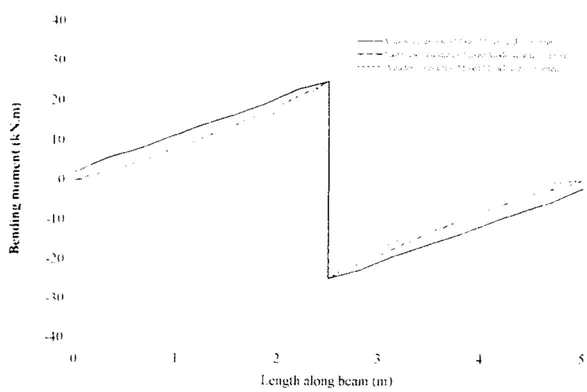

Fig. 6. Bending moment distribution for free beam

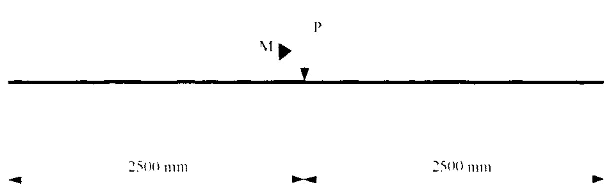

Fig. 7. Tensionless beam with free ends

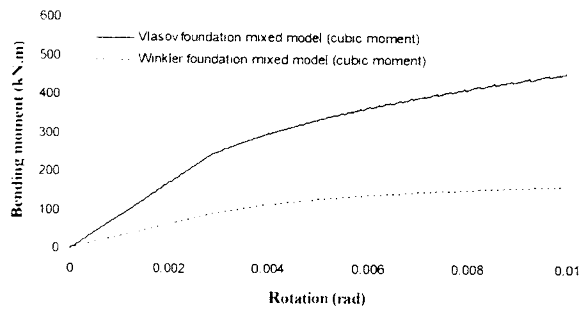

Fig. 8. Tensionless foundation moment-rotation for free beam

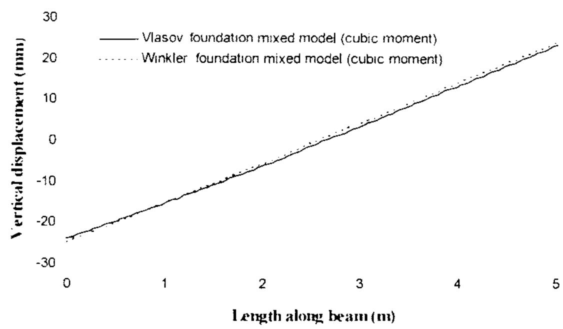

Fig. 9. Tensionless foundation vertical displacement for free beam

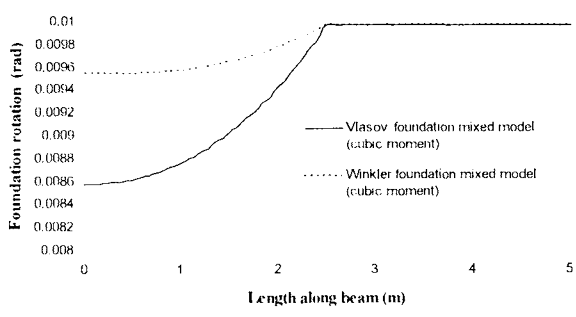

Fig. 10. Tensionless foundation rotation for free beam

5.2. Inelastic Beam on Two-Parameter Foundation

The second numerical example represents an inelastic beam resting on a tensionless foundation. The main objective of this example is to compare the behavior of the Winkler one-parameter model, to the Pasternak and Vlasov two-parameter models. The beam is shown in Fig. (11), and has a length of 10 m, and a square cross-section with 100 mm dimension. For the Pasternak model, an additional soil length that equals twice the beam length on both of its sides was used. The adjacent soil effect on the beam depends on the soil modulus and depth of the soil layer. For the Vlasov model, an additional soil length of twice the foundation length (20 m) was added on both sides of the beam. The beam uniaxial stress-strain relation is elasto-plastic with Young's modulus $E = 200$ GPa, yield strength of 207 MPa, and a hardening slope that equals 1.4%. The beam section is subdivided into 16 fibers. The underlying soil is 10 m Nevada Sand with properties as given by Pradhan and Desai [30] as follows: elastic modulus E_S=40.85 MPa, and Poisson ratio ν_s=0.316.The soil parameters are being calculated with the analytical method proposed by Vallabhan and Das [15] and described in equations (6) to (10).

The loading condition consists of a transverse force and a moment acting at midspan, which is typical of foundation structures. The transverse force equals 70 kN, and is applied under load control, while the moment is applied incrementally under displacement control. The converged midspan moment rotation behavior of the beam is shown in Fig (12) for the mixed model with 32 elements. In the figure, points A, A' and A'' represent yield points for the Vlasov, Pasternak and Winkler models, respectively. Similarly, points B, B' and B'' represent the corresponding points at the ultimate state. The plot reveals that the stiffness is highest for the Vlasov model and is lowest for the Winkler model. The Vlasov model has also a higher yield and ultimate moment capacities, while the Winkler model has the lowest. The same plot is repeated in Fig. (13) using the displacement-based model. The distribution of the local parameters along the length of the beam, namely the bending moment, curvature, vertical displacement, rotation, and foundation forces are shown in Figs. (14-20) at the ultimate load stage. These figures reveal that the displacement model did not achieve convergence, even with 32 elements.

Fig. (14) shows the curvature distributions of the Winkler, Pasternak, and Vlasov models using the mixed formulation, while Fig. (15) shows the same distributions using the displacement formulation. The plots revealed that the mixed model was able to capture the curvature localization near the midspan and accurately predict the maximum curvature. The displacement model, however, failed to capture this behavior as it underestimated the maximum beam curvature value by a factor of 2.3. This is in part due to the assumed displacement shape functions, which can not represent the steep curvature distribution accurately.

As observed in Figs. (16) and (17) the lift-off at the beam ends where the foundation force vanishes is severe for the Winkler model, and is much less for the Pasternak and Vlasov models. In addition, Fig. (17) reveals that the lift-off region is slightly higher for the Pasternak than for the Vlasov model, and that the foundation force is much smaller for the Pasternak than for the Vlasov model. Furthermore, From Fig. (16), due to consideration of the surrounding soil effect, the displacement at midspan is also much lower for the Vlasov and Pasternak models than for the Winkler model. The beam bending moment values along the length are higher for the Vlasov than for the Pasternak and Winkler models, as shown in Fig. (18), due to the additional moment resistance provided by the semi-infinite soil effects. The foundation rotation along the beam length is higher for the Winkler model than for the Pasternak and Vlasov models as shown in Fig. (19). The foundation moment resistance is zero for the Winkler model; while it is higher for the Vlasov model than for the Pasternak model, as shown in Fig. (20).

5.3. Shear Wall Foundation Structure

Numerical analysis using the mixed model was conducted for the Aluminum shear wall foundation structure SSG04-06 tested by Gajan et al. [31] under an increasing lateral load. The footing is 2.8 m x 0.65 m, and has a Young's modulus of 70,000 MPa. The underlying soil is Nevada sand with modulus of elasticity 45 MPa and Poisson ratio 0.4. The yield force of the soil is 1238 kN/m, which is assumed to be at 35% of its bearing capacity. Fig. (21) shows the monotonic envelope of the moment-rotation plot at the middle of the foundation for both the Vlasov and Winkler models, as well as the experimental results. From the figure, it is observed that the Vlasov model is able to predict the behavior reasonably well, while the Winkler model under-predicts the moment capacity of the foundation. Fig. (22) shows the foundation force distribution at the ultimate load. The maximum foundation force equals 1504 kN/m, which exceeds the soil yield force, indicating that the soil has undergone inelastic deformations.

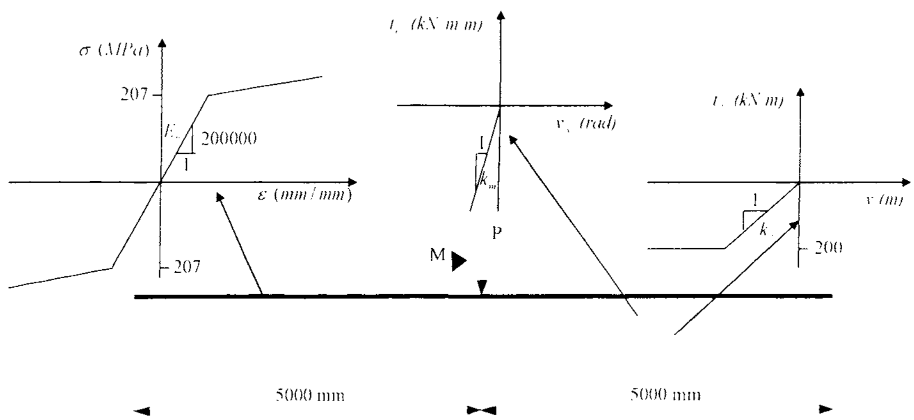

Fig. 11. Inelastic beam on tensionless foundation

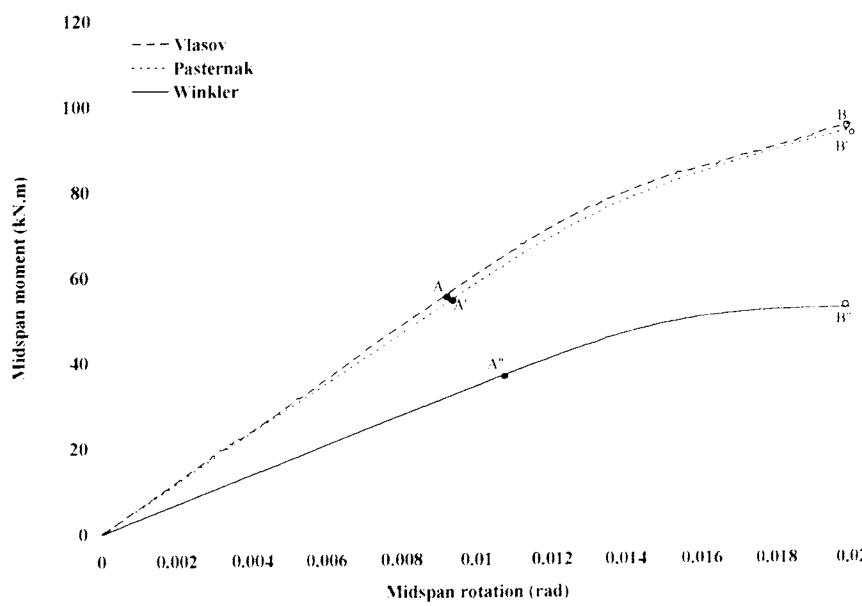

Fig. 12. Moment rotation response of beam on tensionless foundation (mixed model)

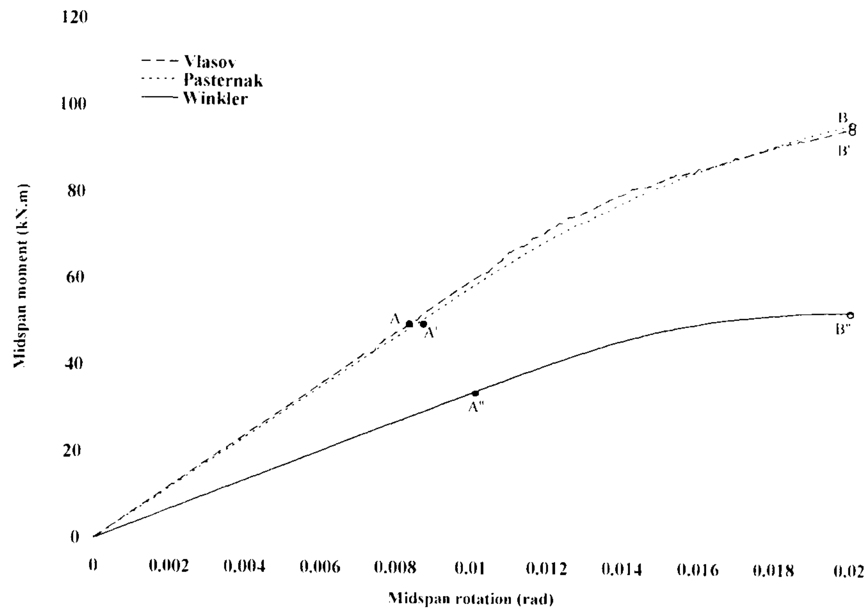

Fig. 13. Moment rotation response of beam on tensionless foundation (displ. model)

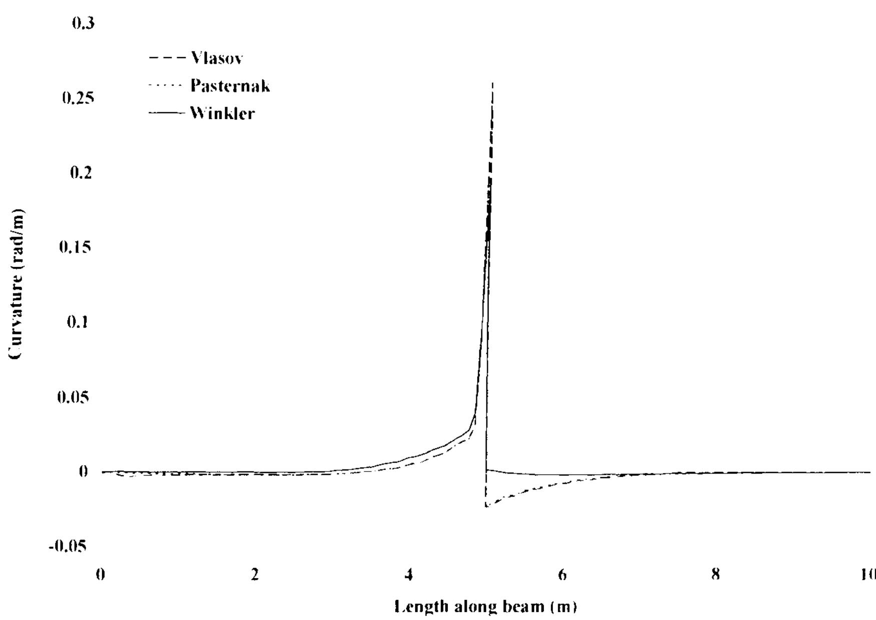

Fig. 14. Curvature distribution for beam on tensionless foundation (mixed model)

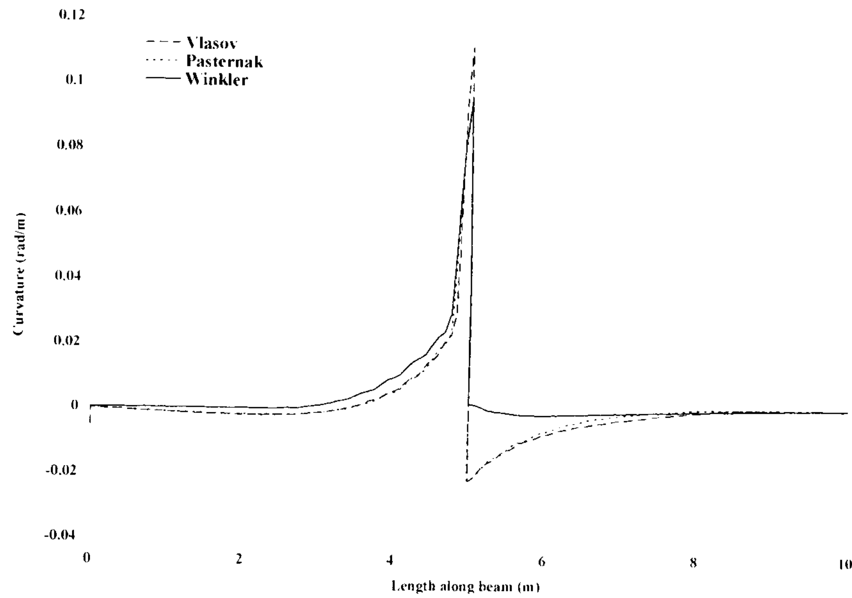

Fig. 15. Curvature distribution for beam on tensionless foundation (displ. model)

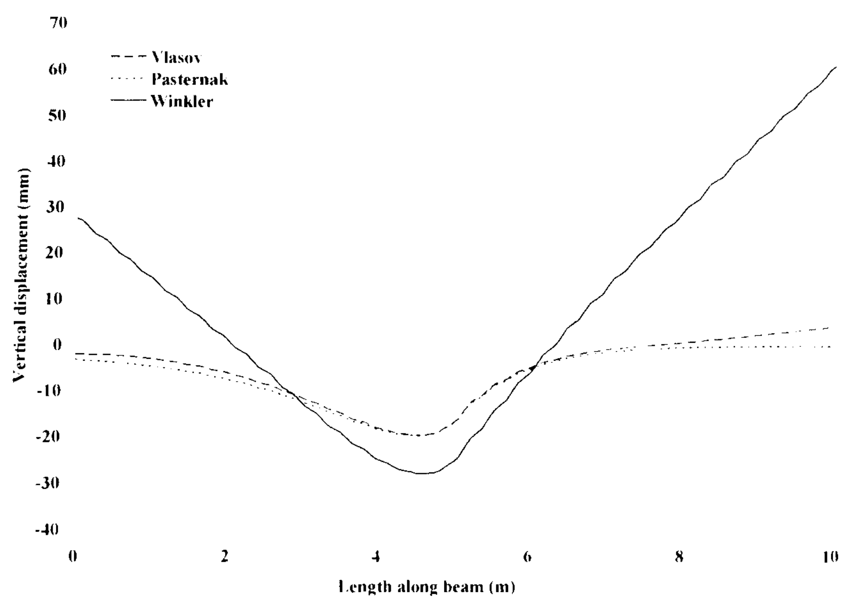

Fig. 16. Vertical displacement distribution for beam on tensionless foundation (mixed model)

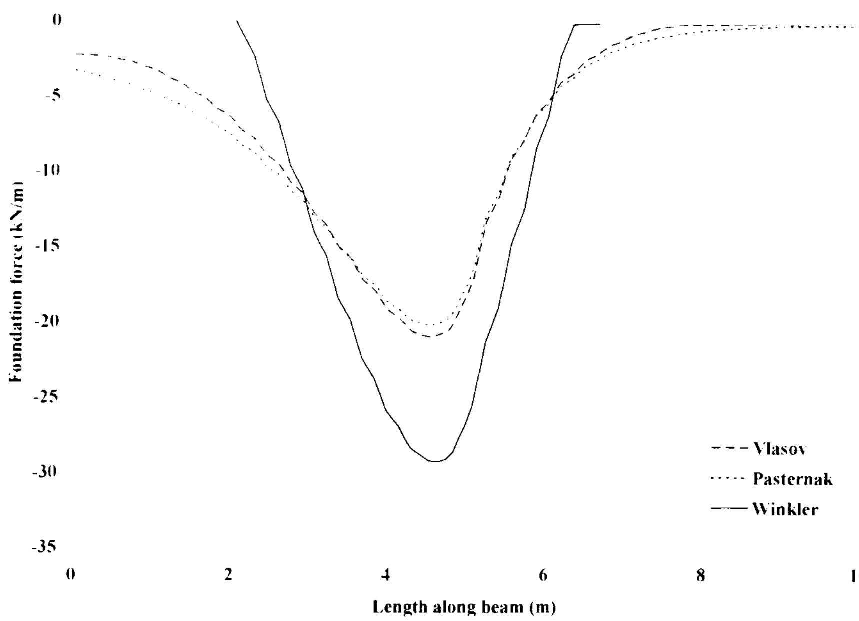

Fig. 17. Foundation force distribution for beam on tensionless foundation (mixed model)

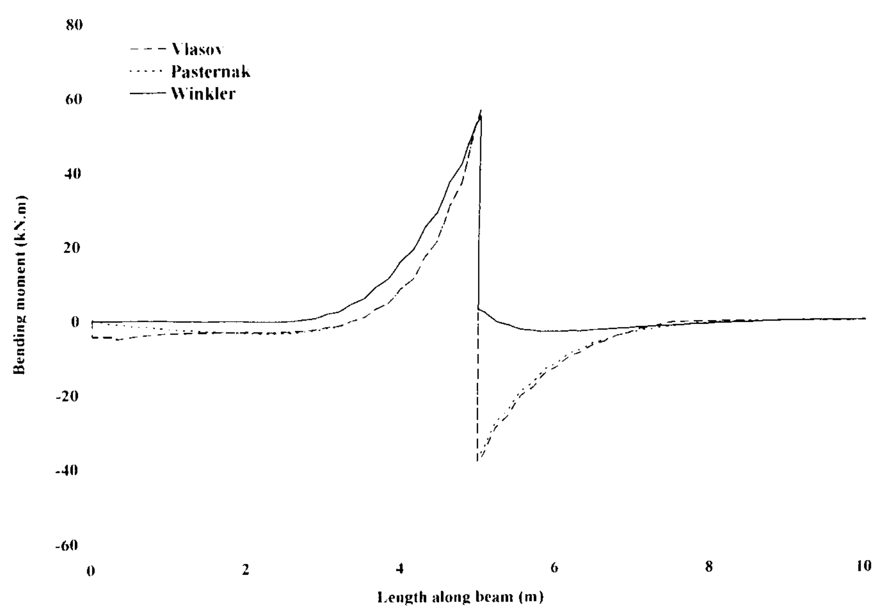

Fig. 18. Moment distribution for beam on tensionless foundation (mixed model)

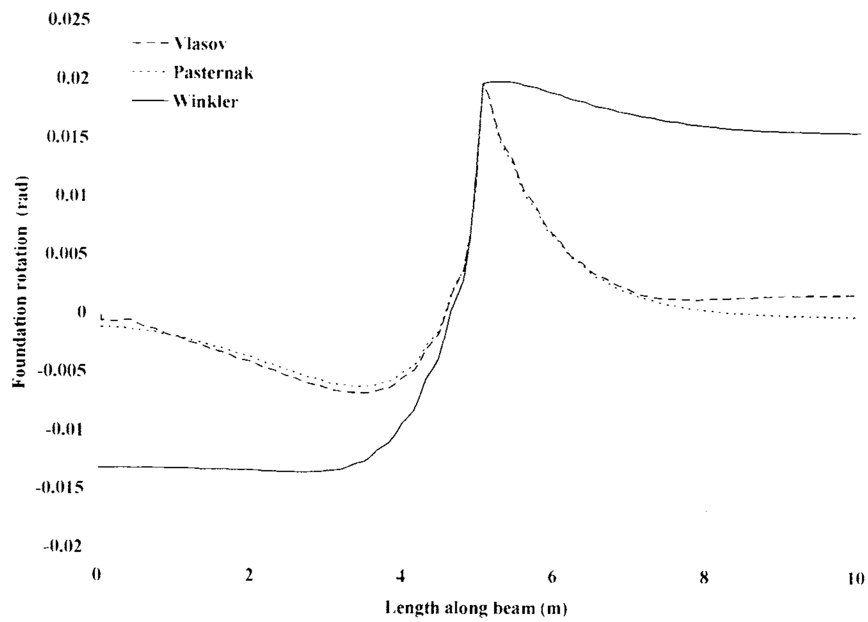

Fig. 19. Foundation rotation for beam on tensionless foundation (mixed model)

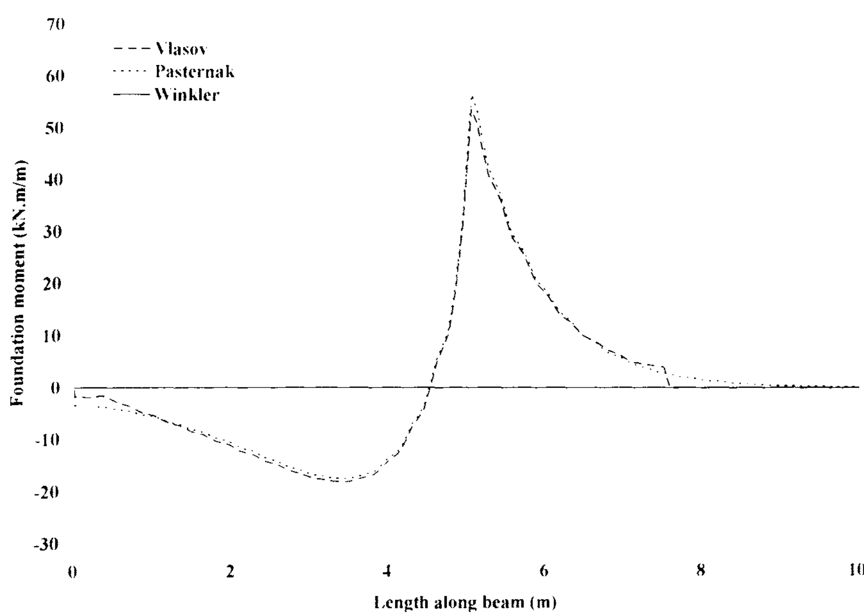

Fig. 20. Foundation moment distribution for beam on tensionless foundation (mixed model)

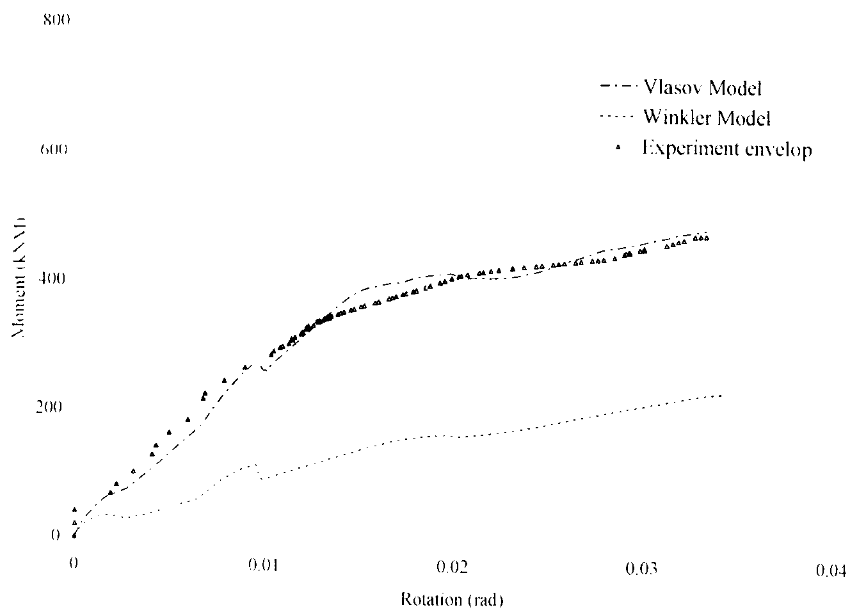

Fig. 21. Moment-rotation response of Gajan et al. specimen

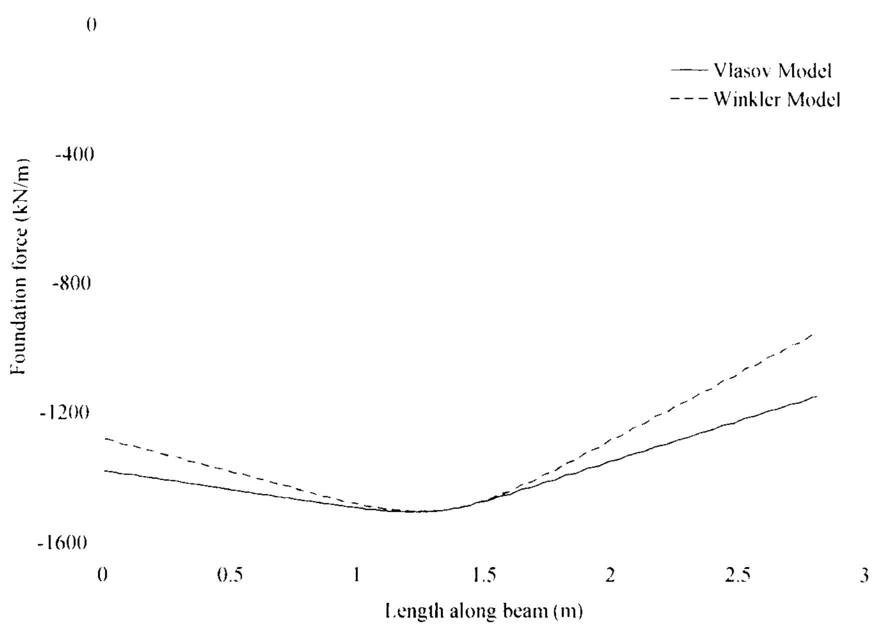

Fig. 22. Foundation force distribution of Gajan et al. specimen

6. Conclusions

The paper presents a new inelastic element for the analysis of two-parameter beam on foundation problems. The element is derived from a two-field mixed formulation, where forces and deformations are approximated with independent interpolation functions. An iterative rational procedure to estimate the values of the two parameters of the foundation based on an assumption of plane strain for the soil medium was presented. This iterative behavior is conducted at each loading step of the nonlinear solution algorithm. The nonlinear response of structures resting on this newly developed two-parameter foundation model is analyzed following both a Vlasov and a Pasternak approach. Numerical examples to compare the behavior of the one-parameter and two-parameter models were conducted. The studies confirmed the importance of including the second parameter in estimating the foundation behavior, and revealed that accounting for the effect of the soil on both sides of the beam by adopting a Vlasov approach can substantially affect the nonlinear response. The studies also confirmed the superiority of the proposed mixed model in evaluating the inelastic complex behavior of these types of structures.

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